

Asymptotical properties of social network dynamics on time scales

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Abstract

In this paper we develop conditions for various types of stability in social networks governed by *Imitation of Success* principle. Considering so-called Prisoner's Dilemma as the base of node-to-node game in the network we obtain well-known Hopfield neural network model. Asymptotic behavior of the original model and dynamic Hopfield model have a certain correspondence. To obtain more general results, we consider Hopfield model dynamic system on time scales. Developed stability conditions combine main parameters of network structure such as network size and maximum relative nodes' degree with the main characteristics of time scale, nodes' inertia and resistance, rate of input-output response.

Keywords: time scale, stability, asymptotical stability, Hopfield neural network, social network, Prisoner's Dilemma

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1. Introduction

A social network is the set of people or groups of people with some pattern of links or interconnection between them. Processes taking place on social networks often may be interpreted as information transition.

The aim of this paper is to consider asymptotic properties of collective opinion formation in social networks with general topology. Transition of opinion between linked nodes will be modelled by game-theoretical mechanism. Total payoff may be a key factor to choose one of the two alternative

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strategies, cooperation or defection in opinion propagation. Such type of dynamics is called *Imitation of Success*. An opposite (in some sense) kind of model is for example the *Voter Model*. Last named model assume imitation of a behavior of uniformly random chosen neighbor node and games payoff has no affects on state updating of the particular node. In paper Manshadi and Saberi (2011) was considered model called *Weak Imitation of Success*. This updating rule is mixture of IS and VM rules: dependently of some parameter ε behavior of updating node may be close to one of the two types of dynamics.

The analysis of the total payoff function for so-called Prisoner's Dilemma leads us to well-known Hopfield neural network model Hopfield (1984). Asymptotic behavior of the direct node-to-node model and dynamic Hopfield model have a certain correspondence. To obtain more general results, we consider Hopfield model on time scale. This problem is discussed in detail in Martynyuk (2012), but we develop more direct and precise conditions for stability of the social network behavior.

2. Preliminary results.

We now present some basic information about time scales according to Bohner and Peterson (2012). A time scale is defined as a nonempty closed subset of the set of real numbers and denoted by \mathbb{T} . The properties of the time scale are determined by the following three functions:

- (i) the forward-jump operator: $\sigma(t) = \inf \{s \in \mathbb{T} : s > t\}$;
- (ii) the backward-jump operator: $\rho(t) = \sup \{s \in \mathbb{T} : s < t\}$ (in this case, we set $\inf \emptyset = \sup \mathbb{T}$ and $\sup \emptyset = \inf \mathbb{T}$);
- (iii) the granularity function $\mu(t) = \sigma(t) - t$.

The behavior of the forward- and backward-jump operators at a given point of the time scale specifies the type of this point. The corresponding classification of points of the time scale is presented in Table 1.

We define a set \mathbb{T}^κ in the following way:

$$\mathbb{T}^\kappa = \begin{cases} \mathbb{T} \setminus \{M\}, & \text{if } \exists \text{ right scattered point } M \in \mathbb{T} : M = \sup \mathbb{T}, \sup \mathbb{T} < \infty \\ \mathbb{T}, & \text{otherwise.} \end{cases}$$

In what follows, we set $[a, b] = \{t \in \mathbb{T} : a \leq t \leq b\}$.

Table 1: Classification of time scale's points

t right-scattered	$t < \sigma(t)$
t right-dense	$t = \sigma(t)$
t left-scattered	$\rho(t) < t$
t left-dense	$\rho(t) = t$
t isolated	$\rho(t) < t < \sigma(t)$
t dense	$\rho(t) = t = \sigma(t)$

Definition 1. Let $f : \mathbb{T} \rightarrow \mathbb{R}$ and $t \in \mathbb{T}^\kappa$. The number $f^\Delta(t)$ is called Δ -derivative of function f at the point t , if $\forall \varepsilon > 0$ there exists a neighborhood U of the point t (i. e., $U = (t - \delta, t + \delta) \cap \mathbb{T}, \delta < 0$) such that

$$|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)| \leq \varepsilon |\sigma(t) - s| \quad \forall s \in U.$$

Definition 2. If $f^\Delta(t)$ exists $\forall t \in \mathbb{T}^\kappa$, then $f : \mathbb{T} \rightarrow \mathbb{R}$ is called Δ -differentiable on \mathbb{T}^κ . The function $f^\Delta(t) : \mathbb{T}^\kappa \rightarrow \mathbb{R}$ is called the delta-derivative of a function f on \mathbb{T}^κ .

If f is differentiable with respect to t then $f(\sigma(t)) = f(t) + \mu(t)f^\Delta(t)$.

Definition 3. The function $f : \mathbb{T} \rightarrow \mathbb{R}$ is called regular if it has finite right limits at all right-dense points of the time scale \mathbb{T} and finite left limits at all points left-dense points of \mathbb{T} .

Definition 4. The function $f : \mathbb{T} \rightarrow \mathbb{R}$ is called rd -continuous if it is continuous at the right-dense points and has finite left limits at the left-dense points. The set of these functions is denoted by $C_{rd} = C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T}; \mathbb{R})$.

The indefinite integral on the time scale takes the form

$$\int f(t) \Delta t = F(t) + C,$$

where C is integration constant and $F(t)$ is the preprimitive for $f(t)$. If the relation $F^\Delta(t) = f(t)$ where $f : \mathbb{T} \rightarrow \mathbb{R}$ is an rd -continuous function, is true for all $t \in \mathbb{T}^\kappa$ then $F(t)$ is called the primitive of the function $f(t)$. If $t_0 \in \mathbb{T}$

then $F(t) = \int_{t_0}^t f(s) \Delta s$ for all t . For all $r, s \in \mathbb{T}$ the definite Δ -integral is defined as follows:

$$\int_r^s f(t) \Delta t = F(s) - F(r).$$

Definition 5. For any regular function $f(t)$ there exists a function F differentiable in the domain D and such that the equality $F^\Delta(t) = f(t)$ holds for all $t \in D$. This function is defined ambiguously. It is called the preprimitive of $f(t)$.

Definition 6. A function $p : \mathbb{T} \rightarrow \mathbb{R}$ is called regressive (positive regressive) if

$$1 + \mu(t)p(t) \neq 0, \quad (1 + \mu(t)p(t) > 0), \quad t \in \mathbb{T}^\kappa.$$

The set of regressive (positive regressive) and rd-continuous functions is denoted by $\mathcal{R} = \mathcal{R}(\mathbb{T})$ ($\mathcal{R}^+ = \mathcal{R}^+(\mathbb{T})$).

Definition 7. For any $p, q \in \mathcal{R}$ by definition put

$$(p \oplus q)(t) = p(t) + q(t) + p(t)q(t)\mu(t), \quad t \in \mathbb{T}^\kappa.$$

It is easy to see that the pair (\mathcal{R}, \oplus) is an Abelian group. As shown in [Bohner Peterson], a function p from the class \mathcal{R} can be associated with a function $e_p(t, t_0)$ which is the unique solution of Cauchy problem

$$y^\Delta = p(t)y, \quad y(t_0) = 1.$$

The function $e_p(t, t_0)$ is an analog, by its properties, of the exponential function defined on \mathbb{R} .

Let us consider dynamic system on time scale \mathbb{T} :

$$\begin{cases} x^\Delta = f(t, x) \\ x(t_0) = x_0. \end{cases} \quad (1)$$

In following formulations we denote solution of (1) by $x(t; t_0, x_0)$.

Definition 8. The equilibrium state $x = x^*$ of the system (1) is uniformly stable if $\forall \varepsilon > 0$ there exists $\delta = \delta(\varepsilon)$, such that

$$\|x_0 - x^*\| < \delta \implies \|x(t; t_0, x_0) - x^*\| < \varepsilon, \quad \forall t \in [t_0, +\infty]_{\mathbb{T}}, t_0 \in \mathbb{T}.$$

Definition 9. *The equilibrium state $x = x^*$ of the system (1) is uniformly asymptotically stable if it is uniformly stable and there exists $\Delta > 0$ such that*

$$\|x_0 - x^*\| < \Delta \implies \lim_{t \rightarrow +\infty} \|x(t; t_0, x_0) - x^*\| = 0, \quad \forall t_0 \in \mathbb{T}.$$

Definition 10. *The equilibrium state $x = x^*$ of the system (1) is uniformly exponentially stable if there exist constants $\alpha, \beta > 0$ ($\beta \in \mathcal{R}^+$) such that*

$$\|x(t, t_0, x_0)\| \leq \|x(t_0)\| \alpha e_{-\beta}(t, t_0), \quad t \geq t_0 \in \mathbb{T},$$

for all $t_0 \in \mathbb{T}$ and $x(t_0) \in \mathbb{R}^n$.

In further definitions and theorems 1 – 3 we assume $f(t, 0) = 0$ for all $t \in \mathbb{T}, t \geq t_0$ and $x_0 = 0$ so that $x = 0$ is a solution to equation (1). For more details see Hoffacker and Tisdell (2005).

Definition 11. *A function $\psi : [0, r] \rightarrow [0, \infty)$ is of class \mathcal{K} if it is well-defined, continuous, and strictly increasing on $[0, r]$ with $\psi(0) = 0$.*

Definition 12. *A continuous function $P : \mathbb{R}^n \rightarrow \mathbb{R}$ with $P(0) = 0$ is called positive definite (negative definite) on D if there exists a function $\psi \in \mathcal{K}$, such that $\psi(\|x\|) \leq P(x)$ ($\psi(\|x\|) \leq -P(x)$) for $x \in D$.*

Definition 13. *A continuous function $P : \mathbb{R}^n \rightarrow \mathbb{R}$ with $P(0) = 0$ is called positive semidefinite (negative semidefinite) on D if $P(x) \geq 0$ ($P(x) \leq 0$) for $x \in D$.*

Definition 14. *A continuous function $Q : [t_0, \infty) \times \mathbb{R}^n \rightarrow \mathbb{R}$ with $Q(t, 0) = 0$ is called positive definite (negative definite) on $[t_0, \infty) \times D$ if there exists a function $\psi \in \mathcal{K}$, such that $\psi(\|x\|) \leq Q(t, x)$ ($\psi(\|x\|) \leq -Q(t, x)$) for $t \in \mathbb{T}, t \geq t_0, x \in D$.*

Definition 15. *A continuous function $Q : [t_0, \infty) \times \mathbb{R}^n \rightarrow \mathbb{R}$ with $Q(t, 0) = 0$ is called positive semidefinite (negative semidefinite) on $[t_0, \infty) \times D$ if $Q(t, x) \geq 0$ ($Q(t, x) \leq 0$) for $t \in \mathbb{T}, t \geq t_0, x \in D$.*

In what follows by $V^\Delta(t, x)$ we denote the full Δ -derivative for function $V(x(t))$ along solution of (1).

Theorem 1. *If there exists a continuously differentiable positive-definite function V in a neighborhood of zero with $V^\Delta(t, x)$ negative semidefinite, then the equilibrium solution $x = 0$ of equation (1) is stable.*

Theorem 2. *If there exists a continuously differentiable, positive definite function V in a neighborhood of zero and there exists a $\xi \in C_{rd}([t_0, \infty); [0, \infty))$ and a $\psi \in \mathcal{K}$, such that*

$$V^\Delta(t, x) \leq -\xi(t)\psi(\|x\|),$$

where

$$\lim_{t \rightarrow \infty} \int_{t_0}^t \xi(s) \Delta s = \infty, \quad (2)$$

then the equilibrium solution $x = 0$ to equation (1) is asymptotically stable.

Theorem 3. *If there exists a continuously differentiable, positive definite function V in a neighborhood of zero and there exists a $\xi \in C_{rd}([t_0, \infty); [0, \infty))$ and a $\psi \in \mathcal{K}$, such that*

$$V^\Delta(t, x) \leq \xi(t)\psi(\|x\|),$$

where (2) holds, then the equilibrium solution $x = 0$ to equation (1) is unstable.

Here and elsewhere we shall use spectral matrix norm as a norm by default:

$$\|A\|_2 = \sqrt{\lambda_{\max}(A^*A)}.$$

Now we formulate a base model for Hopfield network dynamics and few important results about stability of its solutions. Indeed, let us consider dynamic equation of the type

$$x^\Delta(t) = -Bx(t) + Ag(x(t)) + J, \quad (3)$$

where $t \in \mathbb{T}$, $\sup \mathbb{T} = +\infty$, $x(t) \in \mathbb{R}^n$, $A = (a_{ij})$, $i, j = \overline{1, n}$, $B = \text{diag}(b_i)$, $b_i > 0$, $i = \overline{1, n}$, $J = (J_1, \dots, J_n)^T$, $g(x) = (g_1(x_1), \dots, g_n(x_n))^T$. Also, $\bar{b} = \max_i \{b_i\}$, $\underline{b} = \min_i \{b_i\}$. Conceptual meaning of model's components will be clarified below.

We assume on system (3) as follows.

S_1 . The vector-function $f(x) = -Bx + Ag(x) + J$ is regressive.

S_2 . There exist positive constants $M_i > 0$, $i = \overline{1, n}$, such that $|g_i(x)| \leq M_i$ for all $x \in \mathbb{R}$.

S_3 . There exist positive constants $\lambda_i > 0, i = \overline{1, n}$ such that $|g_i(x') - g_i(x'')| \leq \lambda_i |x' - x''|$ for all $x', x'' \in \mathbb{R}$. In what follows we denote $\Lambda = \text{diag}(\lambda_i)$, $L = \max_i \lambda_i$.

Definition 16. An $n \times n$ matrix A that can be expressed in the form $A = sE - B$, where E is an identity matrix, $B = (b_{ij})$ with $b_{ij} \geq 0, 1 \leq i, j \leq n$, and $s \geq \rho(B)$, the maximum of the moduli of the eigenvalues of B , is called an M -matrix.

It should be noted that M -matrix can be characterized in many other ways. Detailed description of forty such ways one can find in Plemmons (1977). For our purpose we find useful the following definition.

Definition 17. An $n \times n$ matrix A with non-negative diagonal elements and non-positive off-diagonal ones is called M -matrix when real part of each eigenvalue of A is positive.

Lemma 1. (Martynyuk, 2012, lemma 5.1.2) Let assumption S_3 be fulfilled. If for every fixed $t \in \mathbb{T}$ the matrix $(I - \mu(t)B)\Lambda^{-1} - \mu(t)|A|$ is an M -matrix, the function $f(x) = -Bx + Ag(x) + J$ is regressive.

Lemma 2. (Martynyuk, 2012, lemma 5.1.1) If for system (3) conditions $S_1 - S_3$ are satisfied then there exists an equilibrium state $x = x^*$ of system (1) and moreover, $\|x^*\| \leq r_0$, where

$$r_0 = \left(\sum_{i=1}^n \frac{1}{b_i^2} \left(\sum_{j=1}^n M_j |a_{ij}| + |J_i| \right)^2 \right)^{\frac{1}{2}}.$$

Besides, if the matrix $B\Lambda^{-1} - |A|$ is an M -matrix, this equilibrium state is unique.

And last result we need is so-called Gershgorin circle theorem. Let A be a complex $n \times n$ matrix, with entries a_{ij} . For $i \in \{1, \dots, n\}$ let ρ_i be the sum of the absolute values of the non-diagonal entries in the i -th row. Let $D(a_{ii}, \rho_i)$ be the closed disc centered at a_{ii} with radius ρ_i . Such a disc is called a Gershgorin disc.

Theorem 4 (Gershgorin circle theorem). Every eigenvalue ν of A lies within at least one of the Gershgorin discs $D(a_{ii}, \rho_i)$, i. e. there exists $i \in \{1, \dots, n\}$ such that

$$|\nu - a_{ii}| \leq \rho_i = \sum_{j \neq i} |a_{ij}|.$$

3. Main results

3.1. Node-to-node game setup.

Let us define a set of nodes $V = \{1, 2, \dots, n\}$. Each member of V is interpreted as player in some matrix game with its neighbors. This game repeats at time steps, discrete or continuous. Set of i -th node neighbors we denote by Ω_i , $k_i = |\Omega_i|$. Here we consider only one type of matrix game is known as Prisoner's Dilemma. Each node has two strategies: cooperate (C) and defect (D). Payoff matrix is illustrated below:

$$P = \begin{pmatrix} b - c & b \\ -c & 0 \end{pmatrix}.$$

Here b is a benefit provided by node to its co-player, c is a cost of cooperation and hereafter we assume $b > c$. In this case the strategy of mutual defection is the only Nash equilibrium, while mutual cooperation is more acceptable social outcome.

Current state of i -th node at moment t we denote by $S_i(t) \in \{0, 1\}$, where zero state represent the defection strategy. Easy to show that at the instant of time t node i gets total payoff equal to $-k_i c S_i(t) + \sum_{j \in \Omega_i} b S_j(t)$. This equation remains correct regardless of the nature of time. Hence in what follows we assume $t \in \mathbb{T}$, where \mathbb{T} is time scale.

3.2. Hopfield network setup.

Assume reaction of each node in network is governed by simple threshold rule:

$$S_i(t) = \begin{cases} 0, & \text{if } -k_i c S_i(t) + \sum_{j \in \Omega_i} b S_j(t) < U_i, \\ 1, & \text{if } -k_i c S_i(t) + \sum_{j \in \Omega_i} b S_j(t) \geq U_i, \end{cases}$$

where U_i is individually payoff threshold for cooperation. With the aim of using Hopfield neurons model we transform last threshold rule to the rule with continuous responses. In the end this transformation will lead us to dynamical system on time scale modelling asymptotic behavior of network.

Let the state variable S_i for i -th "neuron" have the range $[0, 1]$ and be a continuous and strictly increasing function of the total payoff u_i . In biological terms S_i and u_i are output and input signal of i -th "neuron" respectively. Input-output relation we denote by $g_i(u_i)$, so $S_i(t) = g_i(u_i(t))$ and $u_i(t) =$

$g^{-1}(S_i(t))$. If some node having non-zero payoff abruptly loses all connections in the network, its behavior may be described by simple dynamic equation:

$$\begin{cases} C_i u_i^\Delta(t) = -\frac{u_i(t)}{R_i}, \\ u_i(0) = u_{i0}. \end{cases}$$

By analogy with electrical circuit theory in last equation C_i is called capacitance of i -th node and R_i is called its resistance. Obviously, without communication node's payoff will decay to zero as individual intention of cooperation do.

With communication the node gets additional payoff playing with its neighbours, so dynamic equation becomes as follows:

$$C_i u_i^\Delta(t) = -ck_i S_i(t) + \sum_{j \in \Omega_i} b S_j(t) - \frac{u_i(t)}{R_i}, \quad i = \overline{1, n},$$

or, using input-output relation and adjacency matrix of network $D = (d_{ij}), i, j = \overline{1, n}$,

$$C_i u_i^\Delta(t) = -ck_i g_i(u_i) + \sum_j b d_{ij} g_j(u_j) - \frac{u_i}{R_i}, \quad i = \overline{1, n}.$$

Let us introduce two matrices:

$$A = \begin{pmatrix} -\frac{k_1 c}{C_1} & \frac{bd_{12}}{C_1} & \cdots & \frac{bd_{1n}}{C_1} \\ \frac{bd_{21}}{C_2} & -\frac{k_2 c}{C_2} & \cdots & \frac{bd_{2n}}{C_2} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{bd_{n1}}{C_n} & \frac{bd_{n2}}{C_n} & \cdots & -\frac{k_n c}{C_n} \end{pmatrix}, \quad B = \begin{pmatrix} \frac{1}{R_1 C_1} & 0 & \cdots & 0 \\ 0 & \frac{1}{R_2 C_2} & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \frac{1}{R_n C_n} \end{pmatrix}.$$

Then we can rewrite equation in vector form:

$$u^\Delta(t) = -Bu(t) + Ag(u(t)). \quad (4)$$

It is easy to see that without significant changes in arguments we can consider more general model with constant input for every node in network. By denoting this input as vector $J = (J_i)$, we finally get our main equation as follows:

$$u^\Delta(t) = -Bu(t) + Ag(u(t)) + J. \quad (5)$$

3.3. Stability condition for network game

Theorem 5. *For system (5) assume that conditions $S_1 - S_3$ are valid and*

$$k_i < \frac{\lambda_i}{R_i(c+b)}, \quad i = \overline{1, n}.$$

Then there exists unique equilibrium state $u = u^$ of system (5) and $\|u^*\| \leq r_0$.*

Proof. Let us show $Q = B\Lambda^{-1} - |A|$ be an M -matrix. Using inequality for number of neighbors it is easy to derive estimation for real part of eigenvalues ν of matrix Q . Indeed, by Gershgorin circle theorem $\operatorname{Re} \nu > 0$ if and only if $Q_{ii} - \rho_i > 0$ for all $i = \overline{1, n}$, where

$$Q_{ii} = \frac{\lambda_i}{R_i C_i} - \frac{k_i c}{C_i},$$

$$\rho_i = \sum_{j \neq i} |Q_{ij}| = \frac{b}{C_i} \sum_{j \neq i} d_{ij} = \frac{k_i b}{C_i}.$$

We can write

$$Q_{ii} - \rho_i = \frac{\lambda_i}{R_i C_i} - \frac{k_i c}{C_i} - \frac{k_i b}{C_i} = \frac{\lambda_i}{R_i C_i} - k_i \frac{c+b}{C_i} > 0$$

and it is obvious that the upper bound for k_i in theorem's statement does guarantee last inequality. Hence Q is an M -matrix and now to end the proof it remains to apply lemma 2. ■

Remark 1. *It is interesting to notice that existence of unique stable state in network does not depend on nodes' "capacitance".*

Remark 2. *For every particular node in network ratio $\frac{\lambda_i}{R_i}$ describes its potential activity. If $\lambda_i > k_i R_i$ then node's output reaction can conquer with its overall "resistance" to accept neighbor's behaviour.*

Remark 3. *Notice that in theorem key role plays overall payoff scale of the game expressed as sum $b + c$.*

It is particularly remarkable that existence of the unique equilibrium state in network is robust against vanishing of any particular node. Indeed, nodes fulfil conditions of theorem 5 independently. So if there exists the unique

equilibrium state, vanishing any particular node does not affect on the conditions for all other nodes. On the other hand, a new node may easily violate conditions of theorem and break the existence of equilibrium.

In two theorems below we use Lyapunov method to formulate sufficient conditions for asymptotic stability of stable state in network dynamics. Let $u = u^*$ be the unique stable state of (5), i. e.

$$-Bu^* + Ag(u^*) + J = 0.$$

By introducing new variable $z = u - u^*$ we obtain dynamical system on time scale \mathbb{T}

$$z^\Delta(t) = -Bz(t) + Ah(z(t)), \quad (6)$$

where $h(z) = g(z + u^*) - g(u^*)$. If conditions $S_1 - S_3$ are valid for system (5), it is easy to see that

Σ_1 . The vector-function $f(z) = -Bz + Ah(z)$ is regressive.

Σ_2 . $|h_i(z)| \leq 2M_i, i = \overline{1, n}$ for all $z \in \mathbb{R}$.

Σ_3 . $|h_i(z') - h_i(z'')| \leq \lambda_i |z' - z''|, i = \overline{1, n}$ for all $z', z'' \in \mathbb{R}$.

Conditions $\Sigma_1 - \Sigma_3$ guarantee existence and uniqueness for solution of (6) on $t \in [t_0, +\infty)$ for any initial values $z(t_0) = z_0$.

Theorem 6 (Size-dependent condition). *Under the conditions of theorem 5 assume that $\sup \mathbb{T} = +\infty$ and $\mu(t) \leq \mu^*$ for all $t \in \mathbb{T}$. If inequality*

$$\sqrt{n} (b + c) \max_{1 \leq i \leq n} \frac{k_i}{C_i} \leq \frac{-1 - \mu^* \bar{b} + \sqrt{1 + 2\mu^* (\bar{b} + \underline{b})}}{\mu^* L} \quad (7)$$

holds, then unique equilibrium state $u = u^$ of system (5) is uniformly asymptotically stable.*

Proof. Clearly, stability of the trivial solution $z = 0$ of (6) is equivalent to stability of the stable state u^* of (5). Let us choose the $V(z) = z^T z$ as a Lyapunov function. It can easily be checked that $V(z)$ is positive definite. If $z(t)$ is Δ -differentiable at the moment $t \in \mathbb{T}^\kappa$, the full Δ -derivative of $V(z(t))$ along solution of (6) be as follows

$$\begin{aligned} V^\Delta(z(t)) &= (z^T(t)z(t))^\Delta = z^T(t)z^\Delta(t) + [z^T(t)]^\Delta z(\sigma(t)) = \\ &= z^T(t)z^\Delta(t) + [z^T(t)]^\Delta [z(t) + \mu(t)z^\Delta(t)] = \\ &= 2z^T(t)[-Bz(t) + Ah(z(t))] + \mu(t) \|-Bz(t) + Ah(z(t))\|_2^2. \end{aligned}$$

Since B is diagonal matrix with all positive diagonal elements, it follows that maximal eigenvalue of B is $\bar{b} = \max_i \{b_i\}$ and the same one of $-B$ is $\underline{b} = \min_i \{b_i\}$. Using properties of matrix and vector norms and the fact that $\|h(z(t))\| \leq L\|z(t)\|$ it's easy to obtain following estimation:

$$\begin{aligned} V^\Delta(z(t)) &\leq -2\underline{b}\|z(t)\|^2 + 2\|z(t)\| \|A\|_2 \|h(z(t))\| + \mu(t) (\bar{b}\|z(t)\| + \|A\|_2 \|h(z(t))\|)^2 \\ &\leq -2\underline{b}\|z(t)\|^2 + 2L\|A\|_2 \|z(t)\|^2 + \mu(t) (\bar{b}\|z(t)\| + L\|A\|_2 \|z(t)\|)^2 \\ &\leq -\left(2\underline{b} - 2L\|A\|_2 - \mu(t) (\bar{b} + L\|A\|_2)^2\right) \|z(t)\|^2. \end{aligned}$$

It is obvious that $\psi(\|z\|) = \|z\|^2$ belongs to class \mathcal{K} . Let us prove that the function

$$\xi(t) = 2\underline{b} - 2L\|A\|_2 - \mu(t) (\bar{b} + L\|A\|_2)^2$$

under theorems' assumptions belongs to $C_{rd}([t_0, \infty); [0, \infty))$ and fulfills condition (2).

Indeed, we have $\xi(t) \geq 2\underline{b} - 2L\|A\|_2 - \mu^* (\bar{b} + L\|A\|_2)^2$. Hence,

$$\begin{aligned} \lim_{t \rightarrow \infty} \int_{t_0}^t \xi(s) \Delta s &= \lim_{t \rightarrow \infty} \int_{t_0}^t \left(2\underline{b} - 2L\|A\|_2 - \mu(t) (\bar{b} + L\|A\|_2)^2\right) \Delta s \geq \\ &\geq \left(2\underline{b} - 2L\|A\|_2 - \mu^* (\bar{b} + L\|A\|_2)^2\right) \lim_{t \rightarrow \infty} \int_{t_0}^t \Delta s = \infty. \end{aligned}$$

Solving quadratic inequality

$$2\underline{b} - 2L\|A\|_2 - \mu^* (\bar{b} + L\|A\|_2)^2 \geq 0$$

with respect to $\|A\|_2$, we get

$$\frac{-1 - \mu^* \bar{b} - \sqrt{1 + 2\mu^* (\bar{b} + \underline{b})}}{\mu^* L} \leq \|A\|_2 \leq \frac{-1 - \mu^* \bar{b} + \sqrt{1 + 2\mu^* (\bar{b} + \underline{b})}}{\mu^* L}. \quad (8)$$

Clearly, by definition of matrix norm left inequality always holds. We have

$$\begin{aligned}
\|A\|_2 &\leq \sqrt{n} \|A\|_\infty = \sqrt{n} \max_{1 \leq i \leq n} \sum_{j=1}^n |a_{ij}| = \\
&= \sqrt{n} \max_{1 \leq i \leq n} \left\{ \frac{bd_{i1}}{C_i} + \dots + \frac{bd_{i,j-1}}{C_i} + \frac{k_i c}{C_i} + \frac{bd_{i,j+1}}{C_i} + \dots + \frac{bd_{in}}{C_i} \right\} = \\
&= \sqrt{n} \max_{1 \leq i \leq n} \left\{ \frac{k_i c}{C_i} + \frac{b}{C_i} \sum_{j \neq i} d_{ij} \right\} = \\
&= \sqrt{n} \max_{1 \leq i \leq n} \left\{ \frac{k_i c}{C_i} + \frac{k_i b}{C_i} \right\} = \\
&= \sqrt{n} (b + c) \max_{1 \leq i \leq n} \frac{k_i}{C_i}.
\end{aligned}$$

Now it is easy to see that inequality (7) guarantees non-negativity of $\xi(t)$ and condition (2). To conclude the proof, it remains to use theorem 2. ■

Remark 4. We stress that the left side of inequality (7) gathers main parameters of network structure (size and maximum relative nodes' degree). On the other hand, the right side combines main characteristics of time scale, nodes' inertia and resistance, rate of input-output response.

Obviously, for any given matrix A inequality (8) can be checked directly.

Corollary 1. Under the conditions of theorem 5 assume that $\sup \mathbb{T} = +\infty$ and $\mu(t) \leq \mu^*$ for all $t \in \mathbb{T}$. If inequality

$$2\underline{b} - 2L \|A\|_2 - \mu(t) (\bar{b} + L \|A\|_2)^2 \geq 0 \quad (9)$$

holds, then unique equilibrium state $u = u^*$ of system (5) is uniformly asymptotically stable.

Theorem 7 (Size-independent condition). Under the conditions of theorem 5 assume that $\sup \mathbb{T} = +\infty$ and $\mu(t) \leq \mu^*$ for all $t \in \mathbb{T}$. Let C_* denote the minimal "capacitance" in the network and K^* denote the largest node's degree:

$$C_* = \min_{1 \leq i \leq n} C_i, \quad K^* = \max_{1 \leq j \leq n} k_j.$$

If inequality

$$(b+c) \frac{K^*}{C_*} \leq \frac{-1 - \mu^* \bar{b} + \sqrt{1 + 2\mu^* (\bar{b} + \underline{b})}}{\mu^* L} \quad (10)$$

holds, then unique equilibrium state $u = u^*$ of system (5) is uniformly asymptotically stable.

Proof. By repeating the same steps as in previous theorem, we obtain

$$\|A\|_2 \leq \frac{-1 - \mu^* \bar{b} + \sqrt{1 + 2\mu^* (\bar{b} + \underline{b})}}{\mu^* L}.$$

Now if we recall matrix norm inequality $\|A\|_2^2 \leq \|A\|_1 \cdot \|A\|_\infty$, we get

$$\begin{aligned} \|A\|_2^2 &\leq \|A\|_1 \cdot \|A\|_\infty = \\ &= \max_{1 \leq j \leq n} \sum_{i=1}^n |a_{ij}| \cdot (b+c) \max_{1 \leq i \leq n} \frac{k_i}{C_i} \leq \\ &= \max_{1 \leq j \leq n} \left\{ \frac{bd_{1j}}{C_1} + \dots + \frac{bd_{j-1,j}}{C_{j-1}} + \frac{k_j c}{C_j} + \frac{bd_{j+1,j}}{C_{j+1}} + \dots + \frac{bd_{nj}}{C_n} \right\} \cdot (b+c) \max_{1 \leq i \leq n} \frac{k_i}{C_i} \leq \\ &= \max_{1 \leq i \leq n} \left\{ \frac{k_j c}{C_j} + \frac{b}{C_*} \sum_{i \neq j} d_{ij} \right\} \cdot (b+c) \frac{K^*}{C_*} \leq \\ &= \max_{1 \leq i \leq n} \left\{ \frac{k_j c}{C_*} + \frac{bk_j}{C_*} \right\} \cdot (b+c) \frac{K^*}{C_*} \leq \\ &= (b+c) \frac{K^*}{C_*} \cdot (b+c) \frac{K^*}{C_*} = (b+c)^2 \left(\frac{K^*}{C_*} \right)^2. \end{aligned}$$

It is obvious that inequality (10) guarantees non-negativity of $\xi(t)$ and condition (2). To conclude the proof, it remains to use theorem 2. \blacksquare

Remark 5. For large network, i. e. $n \gg 1$, size-dependent condition (7) is unlikely to be fulfilled. In the same time condition (10) can be valid regardless of network's size.

Theorem 8 (Rate of convergence). Under the conditions of theorem 5 assume that $-\underline{b} \in \mathcal{R}^+$ and $\underline{b} - L \|A\|_2 > 0$, where $\underline{b} = \min_i \{b_i\}$. Then

1) solution $z = 0$ of the following system is exponentially stable:

$$z^\Delta(t) = -Bz(t), \quad z(t_0) = z_0, \quad t \geq t_0 \in \mathbb{T}; \quad (11)$$

2) unique equilibrium state $z = 0$ of the system (6) is exponentially stable on $t \geq t_0 \in \mathbb{T}$ and the following estimation holds:

$$\|z(t)\| \leq \|z_0\| \cdot e_{-(\underline{b}-L\|A\|_2)}(t, t_0). \quad (12)$$

Proof. Since $B = \text{diag}(b_i)$ it is easy to obtain fundamental matrix $\Phi_{-B}(t, t_0) = \text{diag}(e_{-b_i}(t, t_0))$.

$$\begin{aligned} \|\Phi_{-B}(t, t_0)\| &= \sqrt{\lambda_{\max}(\Phi_{-B}^T \Phi_{-B})} = \sqrt{\lambda_{\max}(\text{diag}(e_{-b_i}^2(t, t_0)))} = \\ &= \max_{1 \leq i \leq n} |e_{-b_i}(t, t_0)| = e_{-\underline{b}}(t, t_0). \end{aligned}$$

It proofs 1) (see (DaCunha, 2005, Theorem 2.2)).

The solution of (6) satisfies the variation of constants formula Bohner and Peterson (2012)

$$z(t) = \Phi_{-B}(t, t_0)z_0 + \int_{t_0}^t \Phi_{-B}(t, \sigma(s))Ah(z(s)) \Delta s.$$

Hence we have

$$\begin{aligned} \|z(t)\| &\leq \|\Phi_{-B}(t, t_0)z_0\| + \int_{t_0}^t \|\Phi_{-B}(t, \sigma(s)) \cdot Ah(z(s))\| \Delta s \leq \\ &\leq e_{-\underline{b}}(t, t_0)\|z_0\| + \int_{t_0}^t \|e_{-\underline{b}}(t, \sigma(s))\| \cdot \|A\|_2 \|h(z(s))\| \Delta s \leq \\ &\leq e_{-\underline{b}}(t, t_0)\|z_0\| + \int_{t_0}^t \frac{e_{-\underline{b}}(t, s)}{1 - \underline{b}\mu(s)} \cdot \|A\|_2 L \|z(s)\| \Delta s. \end{aligned}$$

Multiplying both sides of inequality by $\frac{1}{e_{-\underline{b}}(t, t_0)} > 0$ (due to $-\underline{b} \in \mathcal{R}^+$) we

obtain

$$\begin{aligned} \frac{\|z(t)\|}{e_{-\underline{b}}(t, t_0)} &\leq \|z_0\| + \int_{t_0}^t \frac{L \|A\|_2}{1 - \underline{b}\mu(s)} \cdot \frac{e_{-\underline{b}}(t, s)}{e_{-\underline{b}}(t, t_0)} \cdot \|z(s)\| \Delta s = \\ &= \|z_0\| + \int_{t_0}^t \frac{L \|A\|_2}{1 - \underline{b}\mu(s)} \cdot \frac{\|z(s)\|}{e_{-\underline{b}}(s, t_0)} \Delta s. \end{aligned}$$

Further, by using Grownall's inequality

$$\frac{\|z(t)\|}{e_{-\underline{b}}(t, t_0)} \leq \|z_0\| \cdot e_{\frac{L\|A\|_2}{1-\underline{b}\mu(s)}}(t, t_0),$$

or

$$\|z(t)\| \leq \|z_0\| \cdot e_{-\underline{b} \oplus \frac{L\|A\|_2}{1-\underline{b}\mu(s)}}(t, t_0) = \|z_0\| \cdot e_{-(\underline{b}-L\|A\|_2)}(t, t_0).$$

By conditions of theorem we have $-(\underline{b} - L\|A\|_2) \in \mathcal{R}^+$. Therefore, the last estimate means that the solution $z = 0$ of (6) is exponentially stable. This completes the proof of theorem. \blacksquare

Proving Theorems 6, 7 we obtain two variants of majorization for $\|A\|_2$. Both of them can be easily used to find direct conditions of exponential stability expressed in the terms of network's structure and nodes' internal properties.

Corollary 2. *Under the conditions of theorem 5 assume that $-\underline{b} \in \mathcal{R}^+$ and*

$$\frac{\underline{b}}{L} > \min \left\{ \sqrt{n} (b + c) \max_{1 \leq i \leq n} \frac{k_i}{C_i}, (b + c) \frac{K^*}{C_*} \right\}.$$

Then unique equilibrium state $z = 0$ of the system (6) is exponentially stable on $t \geq t_0 \in \mathbb{T}$ and the estimation (12) holds.

Obviously, the exponential convergence of the solution for (6) to zero and the solution $u(t)$ for (5) to the unique equilibrium u^* are the same.

4. Discussion.

In this paper we developed conditions for various types of stability in social networks governed by *Imitation of Success* principle. There isn't direct, one-to-one correspondence between considered Hopfield neural network

model and original game-based model. Hence all obtained results can be considered only as the base of understanding of opinion propagation in social network.

We limited ourselves to the one type of node-to-node game, Prisoner's Dilemma. Moreover, arguing we deliberately choose few key elements such as M -matrix characterization, spectral norm estimation etc. Choosing this elements we were guided by the aim to obtain simple, fast-checkable and meaningful conditions.

It is an open problem to study network dynamics based on the another interesting matrix game types. Perhaps, taking into account network's topology or considering particular type of time scale one can develop more specific, precise, and useful results.

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